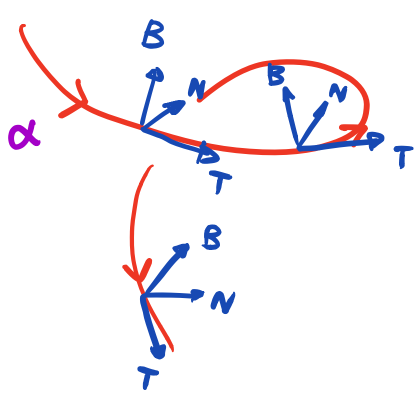
§ Space curves

Consider now a space curve

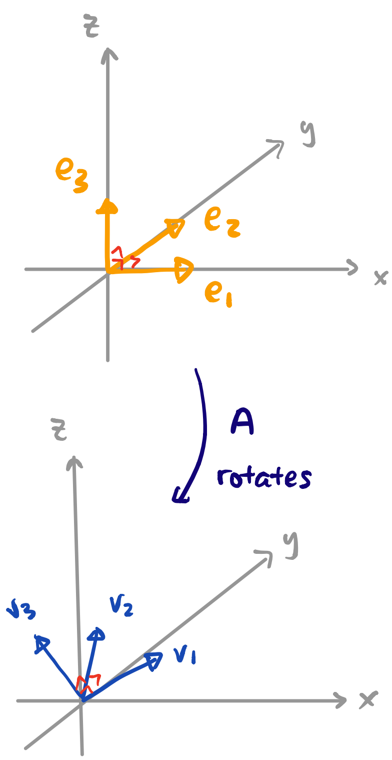
$$\alpha : I \longrightarrow \mathbb{R}^3 \qquad \text{p.b.a.l.}$$



<u>Goal</u>: Define a "moving frame" along α whose rate of change reflects the (extrinsic) "geometry" of α

Frenet frame : $\{T, N, B\}$

Recall first about "frames in $\mathbb{R}^3$"



$\{e_1, e_2, e_3\}$ = standard "frame"

$(e_1 \times e_2 = e_3)$ positively oriented O.N.B.

<u>Linear algebra fact</u>

Given any "frame" $\{v_1, v_2, v_3\}$,

$\exists$ unique $A \in SO(3)$ s.t. $A e_i = v_i$, $i = 1, 2, 3$

$$SO(3) = \{ A \in M_{3\times 3}(\mathbb{R}) : A^T A = I, \det A = 1 \}$$

"Space of frames in $\mathbb{R}^3$"

Recall for plane curves : $\alpha : I \to \mathbb{R}^2$ p.b.a.l.

Frenet frame

$$\{ T , N \}$$

$$\alpha' = T, \quad JT = N$$

Frenet equations

$$\begin{pmatrix} T \\ N \end{pmatrix}' = \begin{pmatrix} 0 & k \\ -k & 0 \end{pmatrix} \begin{pmatrix} T \\ N \end{pmatrix}$$

$k := \langle T', N \rangle$ **Note:** $k = \pm |T'|$

curvature

Now, for a space curve : $\alpha : I \to \mathbb{R}^3$ p.b.a.l.

Define : $T(s) := \alpha'(s)$ tangent

and $\boxed{k(s) := |T'(s)|}$ curvature

Note: $k \geq 0$ for space curves

$\boxed{\text{Assume : } k(s) \neq 0}$ $(*)$

Then, we can define :

$$N(s) := \frac{T'(s)}{|T'(s)|} \qquad \text{normal}$$

and $B(s) := T(s) \times N(s)$ binormal

For any $\alpha : I \to \mathbb{R}^3$ p.b.a.l. satisfying $(*)$ for all $s \in I$, we have defined smoothly along α the

Frenet frame : $\{ T(s) , N(s) , B(s) \}$

Frenet equations: $\alpha : I \to \mathbb{R}^3$ p.b.a.l., $k(s) > 0 \ \forall s \in I$

$$\begin{pmatrix} T \\ N \\ B \end{pmatrix}' = \begin{pmatrix} 0 & k & 0 \\ -k & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix} \quad \ldots\ldots \ (\#)$$

where $\boxed{\begin{aligned} k &:= |T'| \\ \tau &:= \langle B', N \rangle \end{aligned}}$ curvature, torsion

Note: $k \geq 0$ always, but τ can be < 0, $= 0$ or > 0.

Proof of (#): Use $\{T(s), N(s), B(s)\}$ is O.N.B. $\forall s \in I$

Differentiating w.r.t. s:

$$\left.\begin{aligned} \langle T, T \rangle \equiv 1 &\Rightarrow \langle T', T \rangle = 0 \\ \langle N, N \rangle \equiv 1 &\Rightarrow \langle N', N \rangle = 0 \\ \langle B, B \rangle \equiv 1 &\Rightarrow \langle B', B \rangle = 0 \end{aligned}\right\} \Rightarrow \text{diagonal entries in } (\#) = 0.$$

By def$^{\underline{n}}$ $N = \dfrac{T'}{|T'|} = \dfrac{T'}{\underset{\neq 0}{k}} \Rightarrow \boxed{T' = kN}$

By def$^{\underline{n}}$ $B = T \times N \Rightarrow B' = T' \times N + T \times N'$

$$= \underbrace{kN \times N}_{= 0} + \underbrace{T \times N'}_{\perp T}$$

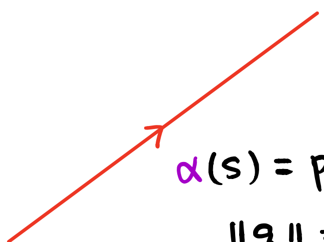
$$\Rightarrow \boxed{B' = \tau N}$$

Finally,

$$\left.\begin{aligned} \langle N, T\rangle \equiv 0 &\Rightarrow \langle N', T\rangle + \overbrace{\langle N, T'\rangle}^{k} = 0 \\ \langle N, B\rangle \equiv 0 &\Rightarrow \langle N', B\rangle + \underbrace{\langle N, B'\rangle}_{\tau} = 0 \end{aligned}\right\} \Rightarrow \boxed{N' = -kT - \tau B}$$

$\square$

(Bad) Example 0: Straight lines



$\alpha(s) = p + sq, \quad s \in \mathbb{R}$

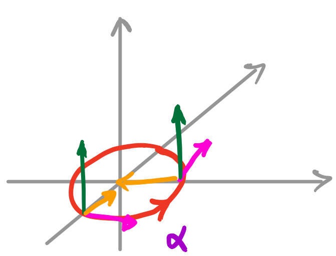
$\|q\| = 1 \quad$ p.b.a.l.

$T(s) = \alpha'(s) = q$

$T'(s) = 0$

$$\boxed{\begin{array}{l} k \equiv 0 \\ \tau \text{ not defined} \end{array}}$$

Example 1: Circles



p.b.a.l.

$\alpha(s) = \left(r\cos\frac{s}{r}, r\sin\frac{s}{r}\right), \quad s \in \mathbb{R}$

$T(s) = \alpha'(s) = \left(-\sin\frac{s}{r}, \cos\frac{s}{r}\right)$

$T'(s) = \frac{1}{r}\left(-\cos\frac{s}{r}, -\sin\frac{s}{r}\right)$

$\therefore \quad k(s) = |T'(s)| = \frac{1}{r} > 0$

$N(s) = \frac{T'(s)}{|T'(s)|} = \left(-\cos\frac{s}{r}, -\sin\frac{s}{r}\right)$

$B(s) = T(s) \times N(s) = (0, 0, 1)$

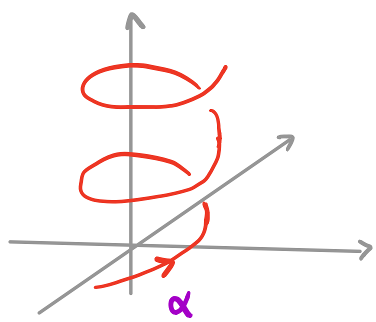
$\therefore \quad \tau(s) = \langle B'(s), N(s)\rangle = 0$

$$\boxed{\begin{array}{l} k \equiv \frac{1}{r} \\ \tau \equiv 0 \end{array}}$$

Exercise: Let $\alpha : I \to \mathbb{R}^3$, p.b.a.l., $k > 0$. Then

"α lies on a plane $P \subseteq \mathbb{R}^3$" $\iff$ $\tau \equiv 0$.

Example 2: Helix



$\leftarrow$ p.b.a.l.

$$\alpha(s) = \left(\cos \frac{s}{\sqrt{2}}, \sin \frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}} \right), \quad s \in \mathbb{R}$$

$$T(s) = \alpha'(s) = \frac{1}{\sqrt{2}} \left(-\sin \frac{s}{\sqrt{2}}, \cos \frac{s}{\sqrt{2}}, 1 \right)$$

$$T'(s) = \frac{1}{2} \left(-\cos \frac{s}{\sqrt{2}}, -\sin \frac{s}{\sqrt{2}}, 0 \right)$$

$$\therefore \; k(s) = |T'(s)| = \frac{1}{2}.$$

$$N(s) = \frac{T'(s)}{|T'(s)|} = \left(-\cos \frac{s}{\sqrt{2}}, -\sin \frac{s}{\sqrt{2}}, 0 \right)$$

$$B(s) = T(s) \times N(s) = \frac{1}{\sqrt{2}} \left(\sin \frac{s}{\sqrt{2}}, -\cos \frac{s}{\sqrt{2}}, 1 \right)$$

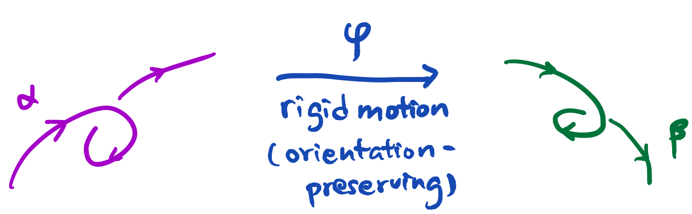
$$B'(s) = \frac{1}{2} \left(\cos \frac{s}{\sqrt{2}}, \sin \frac{s}{\sqrt{2}}, 0 \right)$$

$$\therefore \; \tau(s) = \langle B'(s), N(s) \rangle = -\frac{1}{2}$$

$$\boxed{\begin{aligned} k &\equiv \tfrac{1}{2} \\ \tau &\equiv -\tfrac{1}{2} \end{aligned}}$$

Exercise: k, τ are "geometric" quantity, i.e. they are invariant under orientation-preserving rigid motions of $\mathbb{R}^3$.

$$k_\alpha = k_\beta, \quad \tau_\alpha = \tau_\beta$$



Fundamental Theorem of Space Curves

Given smooth functions $k, \tau : I \to \mathbb{R}$ with $k > 0$, there exists a space curve $\alpha : I \to \mathbb{R}^3$ p.b.a.l. s.t. $k_\alpha = k$ and $\tau_\alpha = \tau$

Moreover, α is unique up to orientation-preserving rigid motions of $\mathbb{R}^3$.

Proof: Fix $s_0 \in I$, and any frame $\{T_0, N_0, B_0\}$ of $\mathbb{R}^3$.

Consider the Frenet equations

$$(\#): \quad \begin{pmatrix} T \\ N \\ B \end{pmatrix}' = \underbrace{\begin{pmatrix} 0 & k & 0 \\ -k & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix}}_{\text{given}} \begin{pmatrix} T \\ N \\ B \end{pmatrix}$$

1^{st} order system linear ODEs

By fundamental existence theorem of ODEs,

$\exists$ solution $T(s), N(s), B(s)$, $s \in I$ to $(\#)$

with "initial condition": $\{T(s_0), N(s_0), B(s_0)\} = \{T_0, N_0, B_0\}$

Claim: $\{T(s), N(s), B(s)\}$ is a frame $\forall s \in I$.

Proof: Define a 3×3 matrix

$$M(s) = \begin{pmatrix} — T(s) — \\ — N(s) — \\ — B(s) — \end{pmatrix}$$

It suffices to check:

$M(s) \in SO(3) \quad \forall s \in I$.

Known: $M(s_0) \in SO(3)$ since $\{T_0, N_0, B_0\}$ is a frame.

Define $Q = MM^T$ (depends on $s \in I$)

Check: Q satisfies the following ODE:

$$(*) \begin{cases} Q' = KQ - QK \\ Q(s_0) = I \end{cases}$$

We can rewrite the Frenet equations (#) in

matrix form: $\boxed{M' = KM}$

where $K = \begin{pmatrix} 0 & k & 0 \\ -k & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix}$ is "skew-symmetric" $\boxed{K^T = -K}$

Therefore,

$$\begin{aligned} Q' &= M'M^T + M(M')^T \\ &= KMM^T + MM^TK^T \\ &= KQ - QK \end{aligned}$$

Note that $Q(s) \equiv I$ is a solution to $(*)$.
By Fundamental uniqueness of ODEs, it must be the only solution. So $MM^T \equiv I$ $\forall s \in I$.
By continuity, $\det M \equiv 1$ $\forall s \in I$, so $M(s) \in SO(3)$ for all $s \in I$. This proves the claim.

Now, once we know

$$\{T(s), N(s), B(s)\} \text{ is a frame } \forall s \in I.$$

We can integrate $\alpha' = T$ to obtain. The rest of the details and the uniqueness part are left as an exercise.

$\square$

Remark on the condition $k > 0$

Otherwise, one may not be able to define the Frenet frame continuously. E.g. consider the trace of a curve

$$\{y = x^3,\ z = 0\} = \text{image}(\alpha)$$